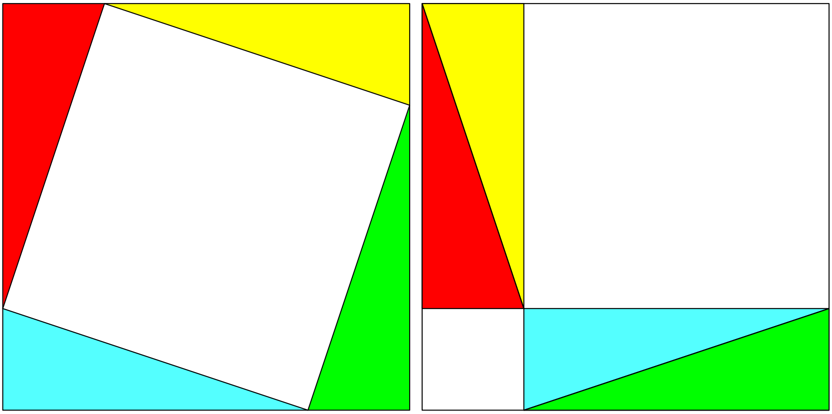
A Proof of Pythagorian's Theorem

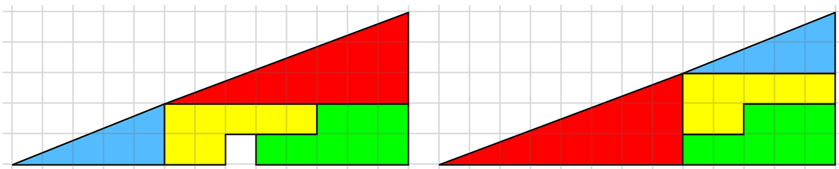


Pythagorian's theorem expresses an equality about the lengths of the sides of a right triangle. It says that if the length of the hypothinus is c and lengths of the other sides are a and b then

$$a^2 + b^2 = c^2.$$

This is proved by the figure above. The large white square on the left is c^2. The two squares on the right are a^2 and b^2.

Take care before accepting such reasoning. Graphical proofs lack formalism and can deceive the casual observer:



$\sqrt{2}$ is irrational

A number is rational if it can be written as a fraction of two integers. If it cannot be, then it is called irrational. For example, 1 is rational because $1 = 1/1$. Since -1.5 can be expressed as $-3/2$ it's also rational.

$\sqrt{2}$ is irrational.

If it were, then one could say that $\sqrt{2} = p/q$ where p and q were integers. p and q can supposed to have no common divisors. (If p and q did have a common divisor, it could be cancled out to produce an irreducible faction.) It follows from $p/q = \sqrt{2}$ that $p = q\sqrt{2}$ and squaring both sides produces $p^2 = 2q^2$.

Thus, p^2 is even because it is 2 times some integer. p must also be even. (The square of an odd number is odd,

$$(2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1,$$

so p cannot be odd.) Since p is even, it can be expressed as $2n$ for some integer n.

$$\sqrt{2} = \frac{p}{q} = \frac{2n}{q}$$

With some algebra, that equation can be manipulated into $q^2 = 2n^2$. As before, q^2 is even because it is 2 times something, and hence q is also even. But that is a contradiction: q and p cannot both be even because they would have a common divisor 2, that they cannot have.

Approximating π

By picking points randomly from a square one can calculuate π. Keep track of how many points from the square you pick lay in the inscribed circle. Supposing the square's edges are length 2, then the inscribed circle has radius 1. Thus the area of the circle is $\pi r^2 = \pi$. The area of the square is 4. The ratio of points picked from the circle to the total number of picked points will converge to $\pi/4$.

This is a C program to demonstrate the convergence.

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>
int main(void) {
  unsigned int total=0, inside=0;
  while(1) {
    float x = (float)rand() / RAND_MAX;
    float y = (float)rand() / RAND_MAX;
    total++;
    if(sqrt(x*x + y*y) < 1) inside++;
    printf("%f\n", 4.0 * inside / total);
  }
}
```

Different Sizes of ∞

There are different sizes of infinity. Small infinites, big infinities, monsterous infinities... The smallest infinity is called a countable infinity. For example, the set of integers, $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, is a countably infinite set. The real numbers, $\mathbf{R}$, which is the the union of the rational numbers and the irrational numbers, is a larger infinity.

At first glance it might seem intuitive that the real numbers is larger size of infinity than the integers. After all, just between the two integers 1 and 2 are infinitely many real numbers! But the concept is subtler: the rational numbers also have infinitely many members between the integers 1 and 2 but the rational numbers and the integers are the same size of infinity–they are both countable.

A set S is countably infinite if $|S| = |\mathbf{Z}|$.

To explain this a formal definition is needed for the meaning of the equals sign in $|\mathbf{Q}| = |\mathbf{Z}|$. For finite sets this is known, count the number of elements in each set – if they have the same count, they are the same size. In infinite sets, if there is a pairing of elements from the first set with elements of the second set, such that all elements have a unique partener in the other set, then the two sets have the same cardinality, the same size.

For example, the set of even integers is the same size as the set of integers. To assert this a pairing is required. Pair each integer n with the even integer $2n$.

$$\begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ -6 & -4 & -2 & 0 & 2 & 4 & 6 \end{array}$$

Because each even integer has a corresponding general integer partener and vice versa, the even integers have the same cardinality as the integers.

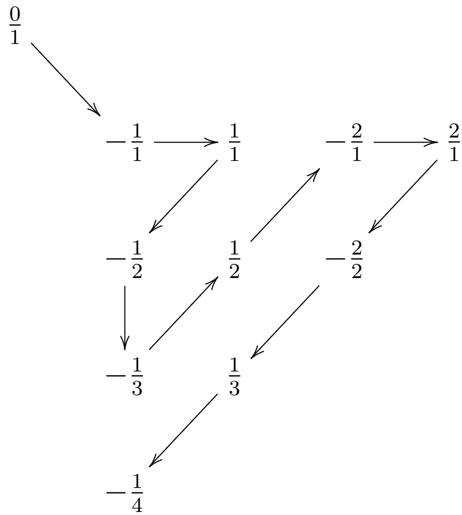
More generally, infinite subsets of countable sets must be countable themselves; this is what was meant by *the smallest infinity*. Because the natural numbers, $\mathbf{N} = \{1, 2, 3, \ldots\}$, are an infinite subset of $\mathbf{Z}$, they are both countable, $|\mathbf{N}| = |\mathbf{Z}|$.

To see that $\mathbf{Q}$ is also countable, pair each integer with each fraction. The pairing can leave no integers without a fraction and no fractions without an integer. Equivelantly pair each fraction with a natural number – count them.

0/1	-1	1	-2	2	-3	3	-4	4 ···
1	-1/1	1/1	-2/1	2/1	-3/1	3/1	-4/1	4/1 ···
2	-1/2	1/2	-2/2	2/2	-3/4	3/2	-4/2	4/2 ···
3	-1/3	1/3	-2/3	2/3	-3/4	3/3	-4/3	4/3 ···
4	-1/4	1/4	-2/4	2/4	-3/4	3/4	-4/4	4/4 ···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

This table provides a means to count the rational numbers, to provide a bijection to $\mathbf{B}$. Every fraction is listed somewhere in this infinitely large table.

The first element of the rational numbers is $0 = 0/1$ in the top left corner of the table. The second element of $\mathbf{Q}$ is just nearby $-1 = -1/1$. Moving right one cell, the thrid element is $1 = 1/1$, then diagonally southwest to the fouth $-1/2$. The fifth (-1/3), the sixith (1/2), and the seventh (-2/1) are found by walking northeast on the diagonal. Moving due east from (-2/1) is the eighth element of the fractions: $2 = 2/1$. The next 3 are found on the same diagonal by moving southwest. And so one snakes up and down the chart, naming off every



rational number and thereby giving pairing with each natural number.

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0/1 & -1/1 & -1/2 & -1/3 & 1/2 & -2/1 & 2/1 \end{array}$$

As the count snakes out across the table, many duplicate rational numbers are seen: $1/1 = 2/2 = 3/3$, $-1/2 = -2/4 = -4/8$, etc. Just skip over duplicate table entries when creating the pairing – otherwise some fractions would be paired with multiple natural numbers.

$|\mathbf{Q}| = |\mathbf{N}| = |\mathbf{Z}|$. The rational numbers are countable. $\square$

Think of the rational numbers as a dust scattered evenly accross the number line. Although pervasive, the dust is ultimately enumerable. The irrational numbers, on the other hand, are a thick paint slabbed across the line. The irrational numbers are more numerous than the rational numbers – even though there are infinitely many rational numbers. The irrational numbers are so numerous that they transcend infinite countability to *uncountability*. No matter how hard one tries, it is impossible to enumerate them.

Suppose the irrational numbers were countable. Then there would be a pairing

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ i_0 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \end{array}$$

Where $i_.$ is irrational.

Gabriel's Horn

Gabriel's Horn is a surface floating in 3 dimensional space. It is formed by taking the graph of the function $r(x) = 1/x$, restricted the domain $x \geq 1$, and spinning it around the x-axis.

The volume of the horn is

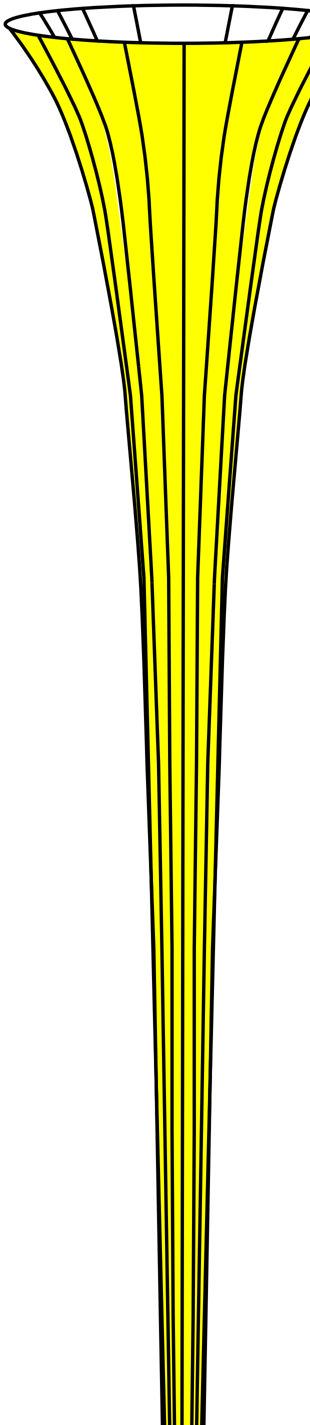
$$\begin{aligned} \int_1^\infty \pi r(x)^2 dx &= -\pi[x^{-1}]_1^\infty \\ &= \pi(1 - \lim_{x\to\infty} x^{-1}) \\ &= \pi \end{aligned}$$

but the surface area is

$$\begin{aligned} \int_1^\infty 2\pi r(x)\sqrt{1 + r'(x)^2}dx &> 2\pi \int_1^\infty r(x)dx \\ &= 2\pi[\ln(x)]_1^\infty \\ &= 2\pi \lim_{x\to\infty} \ln(x) \\ &= \infty. \end{aligned}$$

One could spend a lifetime of fruitless labor attempting to paint the horn but a single glob (of π cubic units) of paint would fill it.

The name references a Christian legend that the archangel Gabriel will blow a horn to announce Judgement Day. For the proper effect, imagine Gabriel sounding the horn from the mouthpeice at heaven ($+\infty$) before racing to our coordinate system to deliver the apocalypse.



Reminder of what continuous means.

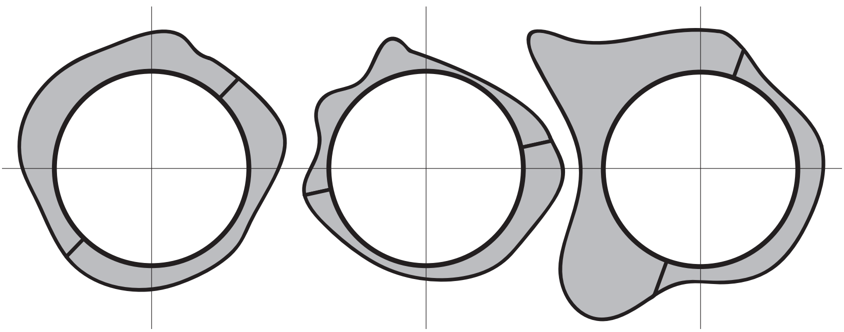
Borsuk-Ulam theorem

At this very instance somewhere over the jungles, mountains, or oceans is a point floating in the atmosphere. On the exact opposite side of the earth, at the same exact instance, is another point. Irrelivant of hurricanes or monsumes these two places have exactly the same tempature and exactly the same atmospheric preasure.

Surprisingly the validity of this claim can be argued without appealing to physical laws but only by using mathematics.

To start we simplfy the assertion to: finding anti-polar points on a circle with the same tempature. Take the smallest circle containg the equater which does not intersect anything but air. The tempature on that circle is continuous (or if not, at least approximated by a continuous function very exactly).

Let C be the circle. (The circle is only the border of the disc– it has no area. Use the unit circle if you want to be formal.) If C has assigned to each of its points a continuously varying real number,



then there are two points, directly opposite of each other, that have the same real number assigned to themselves. In math:

If there is a continuous function $f : C \to \mathbf{R}$, then there exists $x, y \in C$ with $x = -y$ (the minus here means subtraction in two dimensional vectors or complex numbers) and $f(x) = f(-y)$.

Suppose there was not such points. Define a function $g(x) = f(x) - f(-x)$. Because no point on the circle has the same f-value on the opposite side, $g(x) \neq 0$ always. Then pick any random point x on the circle. $f(x) \neq f(-x)$ by assumption, so either $f(x) > f(-x)$ or $f(-x) > f(x)$. The argument will be the same either way–just assume $f(x) > f(-x)$. Thus $g(x) > 0$.

Infinitely many primes

Euler's identity

$e^{\pi i} + 1 = 0$

Monty Hall Problem

Probability of people sharing a birthday

Fermat's little theorem Fermat primality test?

J.W. Alexander's Horned Sphere

Imagine two arms making a chain link with their thumb and index fingers. Neither hand touches the other. But the thumb and index fingers are not normal–nearing the tip the fingers begin to resemble arms. The finger-arm itself has a hand and again creates a link with its partener with the thumb and index finger. Zooming in further these fingers become arms with in turn are linking each other. Continuing on forever one gets the J.W. Alexander's Horned Sphere.

blah blah blah

Combing a Hairy Ball

The Long Line

halting problem? Too well known?

Gödel's incompleteness theorem Too well known?

continuum hypothesis? Too well known?

Circle of polynomials? Too well known?