Problem 1

Let A be a commutative ring with $1 \neq 0$.

(a) If I and J are ideals of A such that $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$ for P a prime ideal of A.

Proof. Suppose $I \nsubseteq P$ and $J \nsubseteq P$. There exists $i \in I$ and $j \in J$ such that $i, j \notin P$. Since P is prime, $ij \notin P$. But $ij \in IJ$. Hence $IJ \nsubseteq P$. □

(b) If the zero ideal is a finite product of maximal ideals

$$0 = M_1 \cdot M_2 \cdots M_n$$

then every prime ideal is maximal.

Proof. Let P be a prime ideal. Suppose for contradiction that P is not maximal, in particular $P \neq M_i$ for $i = 1, 2, \ldots, n$. Hence there exists $m_i \in M_i$ such that $m_i \notin P$. By assumption $m_1 m_2 \cdots m_n = 0$. If we look at that equation modulo P we get

$$(m_1 + P) \cdot (m_2 + P) \cdots (m_n + P) = 0 + P$$

but note A/P is an integral domain and $m_i + P \neq 0 + P$ since $m_i \notin P$. Thus $P = M_i$ for some i. □

Note that I proved the maximal ideas are only the M_i since maximal implies prime.

(c) If A is a commutative Artinian ring then the zero ideal is a finite product of maximal ideals.

Before the proof I will give a small lemma: If I and J are ideals, $IJ \subseteq I$ and $IJ \subseteq J$.

Proof. Let $x \in IJ$. By definition $x = \sum_{k=1}^{n} i_k j_k$ where $n \in \mathbb{N}$, $i_k \in I$ and $j_k \in J$ for $k = 1, 2, \ldots, n$. For any fixed k, $i_k j_k \in I$ since I is an ideal. Since I is closed under addition, $x \in I$. Similarly $x \in J$. □

Proof of Theorem. Let S be the set of ideals which are finite products of maximal ideals. Since every ring has a maximal ideal, S is nonempty. By the Artinian property there exists a minimal element $J = M_1 M_2 \cdots M_n \in S$ where M_i is a maximal ideal.

Help showing $J \subseteq JM$ from Prof. Pakianathan.

Let M be any maximal ideal. By the lemma, $JM \subseteq J$. But since J is a finite product of maximal ideals, JM is a finite product of maximal ideals. But J is minimal in S and $JM \subseteq J$. It must be that $JM = J$.

Now I will show $1-\alpha$ is a unit whenever $\alpha \in J$. Suppose for contradiction that $1-\alpha$ isn't a unit. Then $1-\alpha$ is contained in some maximal ideal M. Since $JM = J$, $\alpha \in JM$ and thus (by the lemma again) $\alpha \in M$. Hence

$$1 = (1-\alpha) + \alpha \in M$$

since M is closed under addition. But that is a contradiction because $1 \in M$ implies $M = (1)$.

Note $J^2 \subseteq J$ by them lemma. But J^2 is also a finite product of maximal ideals, so by the minimality of J, $J^2 = J$.

Suppose for contradiction that $J \neq (0)$. Let $T = \{\text{ideals } I : IJ \neq (0)\}$. T is non-empty because $(1)J \neq (0)$ (there is some non-zero element in J by assumption). By the Artinian property we can find a minimal ideal I such that $IJ \neq (0)$. By the lemma, $IJ \subseteq I$, but $IJJ = IJ^2 = IJ \neq (0)$ so $IJ \in T$ and by the minimality of I, $IJ = I$.

If we take $0 \neq i \in I$, $i = \sum i_k j_k$ where $i_k \in I$ and $j_k \in J$. Since $i \neq 0$ there is some l such that $i_l j_l \neq 0$, let $f = i_l$. Thus $(f)J \neq (0)$ so $(f) \in T$ but $(f) \subseteq I$ so by the minimality of I, $(f) = I - I$ is principle.

Since $IJ = I$ and $f \in I$ there exists $r_k \in A$ and $j_k \in J$ such that $f = \sum (r_k f)(j_k) = f \sum r_k j_k$. Define $\alpha = \sum r_k j_k$, note $\alpha \in J$. We have that $f\alpha = f$ so $0 = f(1-\alpha)$. Since $\alpha \in J$, $1-\alpha$ is a unit. Multiply both sides of $0 = f(1-\alpha)$ by the inverse of $1-\alpha$ and we get that $f = 0$. Hence $I = (0)$ but that contradicts $I \in T$ because $(0)J = (0)$. Our assumption that $J \neq (0)$ must have been false. □

Problem 2

Let R be a ring and M a left R-module. For $m \in M$ the annihilator of m to be

$$Ann_R(M, m) = \{r \in R : rm = 0\}.$$

(b) There is an isomorphism of left R-modules $\langle \alpha \rangle \cong R/Ann_R(M, \alpha)$ for $\alpha \in M$.

Proof. let $f : R \to M$ be given by $f(r) = r\alpha$ where r is acting on α as an element of R. Clearly $\text{img}(f) = \langle \alpha \rangle$ and $\ker(f) = Ann_R(M, \alpha)$. Note $f(r+s) = (r+s)\alpha = r\alpha + s\alpha = f(r) + f(s)$, and $f(rs) = rs\alpha = rf(s)$. Hence f is a R-module homomorphism, thus $\langle \alpha \rangle \cong R/Ann_R(M, \alpha)$. □

(c) Let $V = \mathbb{R}^n$. V is a cyclic left $M_n(\mathbb{R})$-module generated by $e_1 = (1, 0, \ldots, 0)^T$.

Proof. Choose $x = (x_1, x_2, \ldots, x_n)^T \in V$. Let

$$A = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix}.$$

Then clearly $Ae_1 = x$. Hence $V = \langle e_1 \rangle$. $\square$

$Ann_{M_n(\mathbb{R})}(V, e_1)$ = matrices with zeros in the first column. $Ann_{\mathbb{R}}(V, e_1) = \{0\}$, clearly.

(d) $Ann_R(N) = \{r \in R : rn = 0 \ \forall \ n \in N\}$ is a two sided ideal of R for a R-module N.

Proof. Let $a \in Ann_R(N), r \in R$. Note $ran = r0 = 0$ and $arn = an' = 0$ for some $n \in N$. If $b \in Ann_R(N)$ then $(a-b)n = an - bn = 0 - 0 = 0$ so $Ann_R(N)$ is an additive subgroup of R. hence $Ann_R(N)$ is a two-sided ideal of R. $\square$

$$\begin{aligned} Ann_{M_n(\mathbb{R})}(\mathbb{R}^n) &= \{A \in M_n(\mathbb{R}) : Ax = 0 \ \forall \ x \in \mathbb{R}^n\} \\ &= \{0\} \\ Ann_{\mathbb{R}}(\mathbb{R}^n) &= \{r \in \mathbb{R} : r\hat{x} = \hat{0} \ \forall \ \hat{x} \in \mathbb{R}^n\} \\ &= \{0\} \end{aligned}$$

(e) Let I be a left idea l of R. To see that

$$Ann_R(R/I) = \{r \in R : rs \in I \ \forall \ s \in R\},$$

just apply the definition and move a few things around:

$$\begin{aligned} Ann_R(R/I) &= \{r \in R : r \cdot (s + I) = 0 + I \ \forall \ s \in R\}, \\ &= \{r \in R : rs + I = 0 + I \ \forall \ s \in R\}, \\ &= \{r \in R : rs \in I \ \forall \ s \in R\}. \end{aligned}$$

In general $Ann_R(R/I) \subseteq I$ and if R is commutative then there is equality.

Proof. ($\subseteq$) Let $r \in Ann_R(R/I)$, $r = r \cdot 1 \in I$ since $1 \in R$.
($\supseteq$) Assume R is commutative, let $i \in I, s \in R$. Note $i \cdot s = s \cdot i \in I$, $i \in Ann_R(R/I)$. $\square$

If R is not commutative then equality need not hold. For example, $R = M_n(\mathbb{R})$, $I = \{\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) : a, b \in \mathbb{R}\}$. It's easy to see with a little work that I is, in fact, an ideal. Note $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in I$ but since

Counter-example from Justin.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \notin I,$$

$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) \notin Ann_R(R/I)$.

(f) Let N be a R-module. N is also a (well-defined) faithful $R/Ann_R(N)$-module via the action $(r + Ann_R(N)) \cdot n = rn$.

Proof. First I will prove N is a $R/Ann_R(N)$-module by showing the given action is well-defined. Let $r + Ann_R(N) = s + Ann_R(N)$. That is, $r - s \in Ann_R(N)$. Hence $(r - s)n = 0$ for all $n \in N$, but that implies $rn = sn$. So the action is well-defined.

The action is faithful because

$$\begin{aligned} Ann_{R/Ann_R(N)}(N) &= \{\bar{r} \in R/Ann_R(N) : \bar{r} \cdot n = 0 \ \forall\ n \in N\}, \\ &= \{\bar{r} \in R/Ann_R(N) : rn = 0 \ \forall\ n \in N\}, \\ &= \{\bar{r} \in R/Ann_R(N) : r \in Ann_R(N)\}, \\ &= \{0\}. \end{aligned}$$

□

Problem 3

Let R be a ring (not necessarily commutative).

(a) $\mathbb{Z}$-modules are just abelian groups, a $\mathbb{Z}$-submodule will be a normal subgroup. Hence the simple $\mathbb{Z}$-modules are the simple abelian groups.

$\mathbb{R}$-modules are vector spaces over $\mathbb{R}$. The only vector space which does not have a non-trivial subspace is the one dimensional vector space. Hence the only simple $\mathbb{R}$-module is $\mathbb{R}$.

$\mathbb{R}^n$ is a simple $M_n(\mathbb{R})$-modules. Suppose it wasn't – then there exists S a proper non-trivial submodule of $\mathbb{R}^n$. By problem 3 (b), $S = \langle x \rangle$ for some $0 \neq x \in \mathbb{R}^n$. But for any $b \in \mathbb{R}^n$ there exists a matrix $A \in M_n(\mathbb{R})$ such that $Ax = b$ (this is fairly easy to see; I'm too lazy to figure out the exact form of A).

(b) Any simple left R-module is cyclic.

Proof. Let M be a simple left R-module which isn't cyclic. Choose any $0 \neq m \in M$. $\langle m \rangle \neq M$ so $\langle m \rangle$ is a proper nontrivial submodule. □

Let S be a simple left R-module. Then

$$S \cong R/Ann_R(S, \alpha)$$

for any $0 \neq \alpha \in S$ where S is a simple R-module.

Proof. Since S is simple $\langle \alpha \rangle = S$ by the above argument. By problem 2 (b), $S = \langle \alpha \rangle \cong R/Ann_R(S, \alpha)$. □

$Ann_R(S, \alpha)$ is a maximal left ideal of R.

Proof. $R/Ann_R(S,\alpha)$ as an R-module is simple (it's isomorphic to S). But that implies that it cannot have any proper nontrivial left ideals as a ring (ideals are additive subgroups). Hence $R/Ann_R(S,\alpha)$ is a division ring, so $Ann_R(S,\alpha)$ is maximal. □

(c) If M is a maximal left ideal of R then R/M is a simple left R-module.

Proof. R/M is a division ring so there are not any proper nontrivial left ideals of R/M. As a R-module, the left R-submodules are the left ideals of R/M. Hence R/M is simple. □

(d) Define the left Jacobson radical $J_L(R)$ as the intersection of all the left maximal ideals of R.

$J_L(R)$ is equal to the intersection of the annihilators of all left simple R-modules,

$$X = \bigcap_{\text{Simple left } R\text{-mod S}} Ann_R(S).$$

Proof. Note that

$$Ann_R(S) = \bigcap_{0\neq\alpha\in S} Ann_R(S,\alpha)$$

In 3 (b), I proved that each $Ann_R(S,\alpha)$ is a left maximal. Hence X is an intersection of some of the left maximal ideals so $X \supseteq J_L(R)$.

From 3 (c) we know R/M is a simple left ideal for M a maximal left ideal of R. Since M is maximal $1 \notin M$ (otherwise $r \cdot 1 \in M$ for all $r \in R$). Thus $1 + M \neq 0 + M$ in R/M. Note

Help on $J_L(R) \subseteq X$ from Tao.

$$\begin{aligned} Ann_R(R/M, 1+M) &= \{r \in R : r(1+M) = 0 + M\}, \\ &= \{r \in R : r + M = 0 + M\}, \\ &= M. \end{aligned}$$

Also

$$M = Ann_R(R/M, 1+M) \supseteq \bigcap_{r\notin M} Ann_R(R/M, r+M) = Ann_R(R/M).$$

So

$$\begin{aligned} J_L(R) = \bigcap M &= \bigcap Ann_R(R/M, 1+M), \\ &\supseteq \bigcap_{M \text{ maximal}, r\notin M} Ann_R(R/M, r+M), \\ &= \bigcap_{M\text{maximal}} Ann_R(R/M), \\ &\supseteq \bigcap_{S \text{ a simple } R\text{-mod}} Ann_R(S) = X. \end{aligned}$$

Since $\{R/M : M \text{ maximal}\}$ is a subset of the simple left R-modules by 3 (c). □

In 2 (d) we showed $Ann_R(N)$ is a two-sided ideal. Hence $\bigcap_S Ann_R(S)$ is a two-sided ideal.

(e) $\alpha \in J_L(R)$ if and only if $1 - r\alpha$ has a left inverse in R for all $r \in R$.

Proof. Suppose $\alpha \in J_L(R)$ and $1 - r\alpha$ does not have a left inverse for some $r \in R$. Hence $1 - r\alpha$ is in some left maximal ideal M. Since $\alpha \in J_L(R)$, $\alpha \in M$, and $r\alpha \in M$. Thus $r\alpha + (1 - r\alpha) = 1 \in M$ – Contradiction.

On the other hand, suppose $\alpha \notin J_L(R)$. There exists some maximal left ideal M such that $\alpha \notin M$. The ideal generated by elements in M and α contains M, hence is (1) by the maximality of M. Thus there is $m \in M$ and $r, s \in R$ such that $1 = rm + s\alpha$ or $1 - s\alpha = rm \in M$. Any element of M cannot have a left inverse. In conclusion, there exists an element $s \in R$ such that $1 - s\alpha$ does not have a left inverse. □

(f) Let $\alpha \in J_L(R)$, $r \in R$. By (d), $1 - r\alpha$ has a left inverse x_r. So

$$1 = x_r(1 - r\alpha) = x_r - x_r r\alpha$$

Hence $x_r = 1 - x_r r\alpha$. Define $t_r = x_r r\alpha$. Since $J_L(R)$ is an ideal, $t_r \in J_L(R)$. So by problem 3 (e), $1 - t_r$ has a left inverse, but the left and right inverses of $1 - t_r$ must agree. Hence both $1 - t_r$ and $1 - r\alpha$ are units.

$J_L(R) = \{\alpha \in R : 1 - r\alpha \text{ is a unit of } R \ \forall\ r \in R\} =: X$.

Proof. We already know $J_L(R) \subseteq X$ by the above arguments. So let $\alpha \in X$, $r \in R$. So $1 - r\alpha$ is a unit, but that means it has a left inverse. Thus $\alpha \in J_L(R)$ by problem 3 (d). □

$J_L(R) = \{\alpha \in R : 1 - r\alpha s \text{ is a unit for all } r, s \in R\}$.

Proof. If $\alpha \in J_L(R)$ then $\alpha s \in J_L(R)$ so $1 - r\alpha s$ is a unit. Conversely, if $1 - r\alpha s$ is a unit then $\alpha s \in J_L(R)$ for all s, in particular $s = 1$. □

(g) $J(\mathbb{Z}) = (0)$ – the maximal ideals of $\mathbb{Z}$ are $p\mathbb{Z}$.

$J(\mathbb{Z}/n\mathbb{Z}) = (p_1 p_2 \cdots p_s)$ if $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ where each $r_i \geq 1$ and p_i is prime.

Help from Justin finding the Jacobson radicals.

$J(k[x]) \subseteq \bigcap_{\alpha \in k}(x - \alpha)$ since $k[x]/(x - \alpha) \cong k$ by the evaluation homomorphism at k. If $|k| = \infty$ then the intersection of the $(x - \alpha)$ is (0) so $J(k[x]) = (0)$.

$J(k[[x]])$ – ?

$J(M_n(\mathbb{R}))$ – ?

(h) If S is a simple R-module, then S is naturally a simple $R/J(R)$-module.

Proof. Suppose $p + J(R) = q + J(R)$ in $R/J(R)$. Then $p - q \in J(R)$ so $p - q \in Ann_R(S)$. Thus $(p - q)s = ps - qs = 0$. So

$$p + J(R) \cdot s = ps = qs = q + J(R) \cdot s.$$

S is hence a $R/J(R)$-module.

Suppose S was not a simple $R/J(R)$-module. Then S would have a proper nontrivial $R/J(R)$-submodule T. T is an R-module if for $p + J(R) \in R/J(R)$, $t \in T$ we define

$$(p + J(R)) \cdot t = pt$$

for all $p \in R$. Using the same arguments above we see this action is well-defined. So excluding the easy checks we see T is also an R-module. Thus S is not a simple R-module. □

(i) If A is a commutative Artinian ring then $A/J(A)$ is isomorphic to a finite direct product of fields.

Proof. A has only finitely many maximal ideals by problem 1 (c). Call them $M_1, M_2, \ldots, M_n$. Define a map

$$f : A/\bigcap_{i=1}^{n} M_i \to A/M_1 \times A/M_2 \times \cdots \times A/M_n$$

given by

$$f(m + J(A)) = (m + M_1, m + M_2, \ldots, m + M_3).$$

I will prove f is an isomorphism of rings.

Well-defined. Suppose $m + J(A) = n + J(A)$. Then $m - n \in J(A)$ and hence $m - n \in M_i$ for each i. So $f(m + J(A)) = f(n + J(A)$.

Bijection. Let $(m_i + M_i)_{i=1}^n \in \prod_{i=1}^n M_i$. By the Chinese remainder theorem there exists $a \in A$ such that $a \equiv m_i \pmod{M_i}$ for each i and furthermore a is unique modulo $\bigcap M_i = J(A)$. That is, $f(a + J(A)) = (m_i + M_i)_{i=1}^n$. The existence of $a + J(A)$ says f is a surjection, the uniqueness says f is an injection.

Ring Homomorphism. This is a trivial check. □